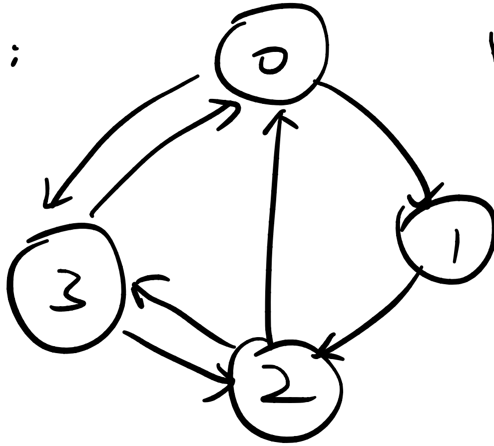
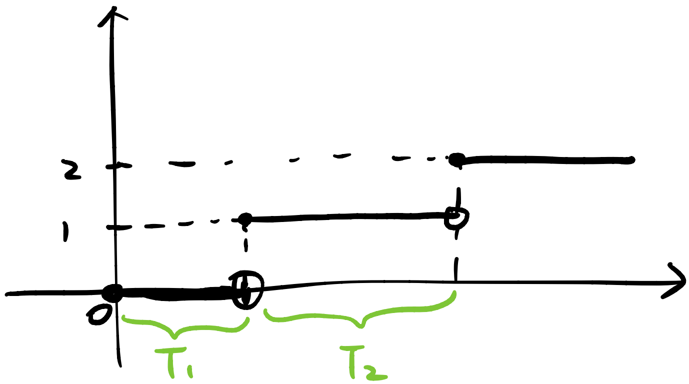
Idea: $\{X_n\}$:



$\forall i,\ P_{ii} = 0$

$\{N_t\}$:



$T_1, T_2, \cdots \overset{i.i.d.}{\sim} \mathcal{E}(\lambda)$

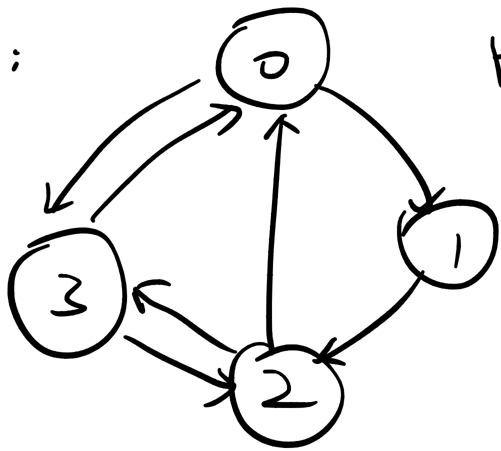
$$X_0 = 0, \quad N_0 = 0, \quad Y_0 = X_{N_0} = X_0 = 0.$$

$\forall t \in (0, T_1)$, $Y_t = X_{N_t} = X_0 = 0$.

$Y_{T_1} = X_{N_{T_1}} = X_1$ may be 1 or 3.

$\forall t \in (T_1, T_1 + T_2)$, $Y_t = X_1$.

$\{X_n\}$:



$\forall i, P_{ii} = 0$

Stay longer at 0, stay shorter at 3.

<u>Holding rate:</u> $q: S \longrightarrow \mathbb{R}_+$

$q(1) = q(2) = 1$, $q(0) = \frac{1}{2}$, $q(3) = 2$.

$E_1, E_2, \cdots \overset{i.i.d.}{\sim} \mathcal{E}(1)$

$Y_0 = 0$. $T_1 = \frac{E_1}{q(0)}$, $\mathbb{E}T_1 = 2$, $T_1 \sim \mathcal{E}(\frac{1}{2})$

$Y_{T_1} = 3$ $T_2 = \frac{E_2}{q(3)}$, $\mathbb{E}T_2 = \frac{1}{2}$, $T_2 \sim \mathcal{E}(2)$

$Y_{T_2} = 2$ $T_3 = \frac{E_3}{q(2)} \sim \mathcal{E}(1)$

BDC: $\{Y_t\}$, with holding rates $q(\cdot)$

If $Y_t = i$, then holding time $\sim \mathcal{E}(q(i))$

After the holding time, state transition happens

$$\begin{tikzcd} & \boxed{i+1} \\ \boxed{i} \arrow[ur, "P_{i,i-1}"] \arrow[dr, "P_{i,i+1}"'] & \\ & \boxed{i-1} \end{tikzcd}$$

, P is transition prob of $\{X_n\}$.

$\begin{cases} B_i = \text{time until next birth at state } i \\ D_i = \text{———————— death ————————} \end{cases}$

<u>Poisson thinning</u>:

$$\begin{tikzcd} & \text{— PP intensity } q(i)\cdot P_{i,i+1} \\ \cdot \arrow[r, "\text{arrival}"'] & \cdot \arrow[u, "\text{if it's birth}"] \arrow[d, "\text{if it's death}"'] \\ & \text{— PP intensity } q(i)\cdot P_{i,i-1} \end{tikzcd}$$

(state transition happens)
intensity $q(i)$

independent

Have proved: B_i indep of D_i,

$$B_i \sim \mathcal{E}(\underbrace{q(i)\cdot P_{i,i+1}}_{\substack{\lambda_i \\ \text{birth rate}}}), \quad D_i \sim \mathcal{E}(\underbrace{q(i)\cdot P_{i,i-1}}_{\substack{\mu_i \\ \text{death rate}}})$$

Rel:

$$\begin{cases} \lambda_i = q(i) \cdot P_{i,i+1} \\ \mu_i = q(i) \cdot P_{i,i-1} \\ \lambda_i + \mu_i = q(i) \end{cases}$$

Identify:

PP $\Longrightarrow$ cts-time BDC (pure-birth)

$$\begin{cases} \lambda_i = \lambda \\ \mu_i \equiv 0 \end{cases}$$

λ_i does not change as i change

(constant-rate pure-birth BDC)